Partial order relation → A relation R on a set A is said to be Partial order relation if it satisfies the following

1) Reflexivity : $aRa \ \forall a \in A$
2) Anti-Symmetric : $aRb \wedge bRa \Rightarrow a = b$
3) Transitive : $aRb \wedge bRc \Rightarrow aRc$

Ex. 1 The relation '|' or 'divisor of' $a|b$, i.e "a is a divisior of b" $\forall a, b \in N$

Ex. 2 The relation "set inclusion" ($\subseteq$) in the set of sets

Ex. 3 The relation '$\leq$' in the usual sence "less than or equal to" on R

Partial order set [POSET]

A set A with a partial ordering relation $\leq$ on A is called a partially ordered set.

Notation → it is denoted by $(A, \leq)$

Ex. $(R, \leq)$ is partially order set (POSET)

Total order relation → Let $(A, \leq)$ be a poset the elements a and b of A are said to be comparable if $a \leq b$ or $b \geq a$

Thus a and b are non comparable if neither $a \leq b$ nor $b \leq a$

Extremal elements of Partially ordered set.

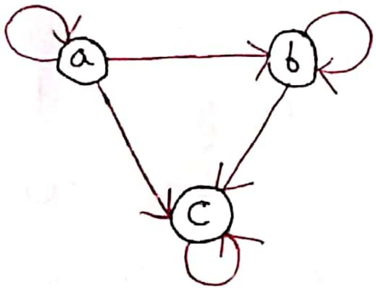
Let $(A, \leq)$ be a POSET. such that if any element $b \in A$ such that $b \leq a$ $(bRa) \forall a \in A$ then b is 'least element' of set A

thus if any element $b \in A$ suchthat $a \leq b$ $(aRb) \forall a \in A$ then b is greatest element of set A

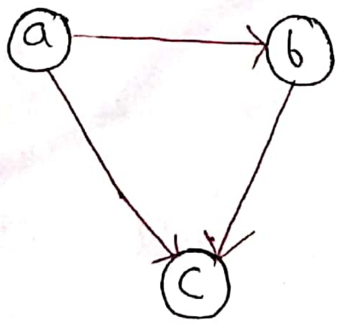
Remarks → Not necessarily a POSET may have Extremal element.

Hasse diagrams → A partial ordering $\leq$ on a POSET A can be represented by a diagram known as a hasse diagram.

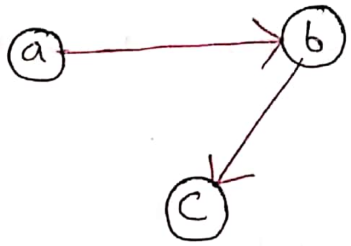
Step-I draw a directed graph



Step-II Remove self loop (Reflexive)



Step - III Remove transitive edge in graph



Step - IV draw diagram bottom to top

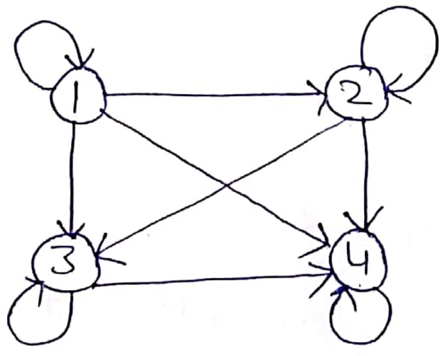
Hasse diagrams.

Ex. 1 Draw a Hasse diagram of the poset $(A, \leq)$

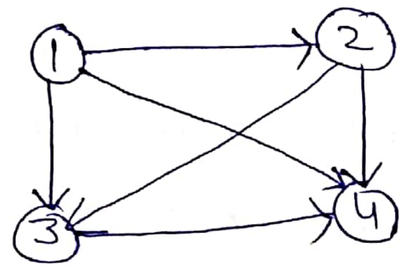
$A = \{1, 2, 3, 4\}$ $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

Sol^n.:

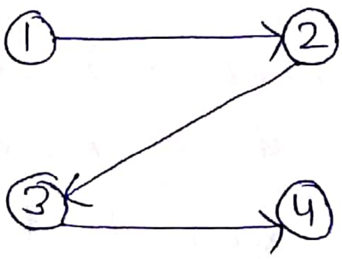
Step - I



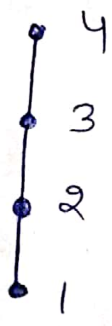
Step - II



Step - III



Step - IV

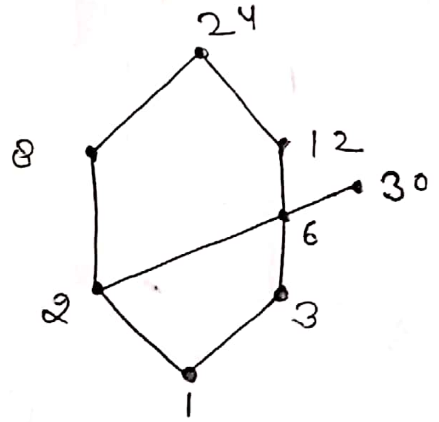


Hasse diagram

Q2 Let $A = \{1, 2, 3, 6, 8, 12, 24, 30\}$ be a set and the relation $\leq$ be defined as $a \leq b$ iff a divides b ; $a, b \in A$ then draw a Hasse diagram of the poset $(A, \leq)$

Sol^n Partial order relation

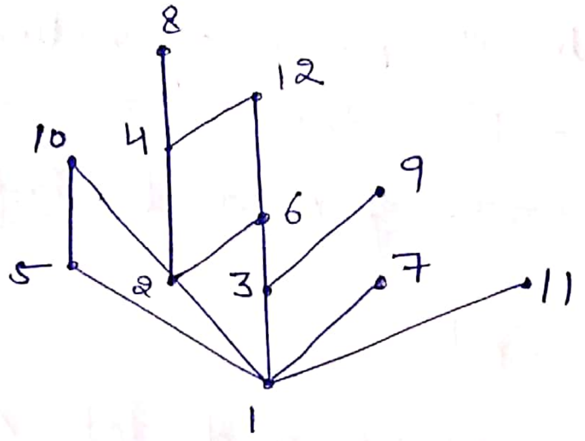
$\leq = \{(1,1), (1,2), (1,3), (1,6), (1,8), (1,12), (1,24)$
$(1,30), (2,2), (2,6), (2,8), (2,12), (2,24)$
$(2,30), (3,3), (3,6), (3,12), (3,24), (3,30)$
$(6,6), (6,12), (6,24), (6,30), (8,8), (8,24)$
$(12,12), (12,24), (24,24), (30,30)\}$



Maximal and minimal Elements →

Let $(A, \leq)$ be a POSET. An element a in A is called a maximal element of A if there is no element $b \in A$ such that $b \neq a$ and $a \leq b$.

An element a in A is called a minimal element of A if there is no element $b \in A$ such that $b \neq a$ and $b \leq a$



maximal element - 7, 8, 9, 10, 11, 12

minimal element — 1

upper bounds and lower bounds :

Let $(P, \leq)$ be a poset and let A be a subset of P. An element $x \in P$ is called an upper bound of A if $a \leq x \ \forall a \in A$

Ex. Let $X = \{1, 2, 3\}$. then $(P(X), \subseteq)$ is a poset

Let $A = \{\phi, \{1, 2\}, \{2\}, \{3\}\}$

then $\{1, 2, 3\}$ is an upper bound of A because every element of A is contained in $\{1, 2, 3\}$

lower bounds : Let $(P, \leq)$ be a poset and let A be a subset of P. An element $x \in P$ is said to be a lower bound of A if $x \leq a \ \forall a \in A$

ex. Let $(N, \leq)$ be the poset of natural no.
Let $A = \{5, 7, 9\}$ then 1, 2, 3, 4 and 5 are lower bounds of A and $\inf(A) = 5$

Least upper bound:

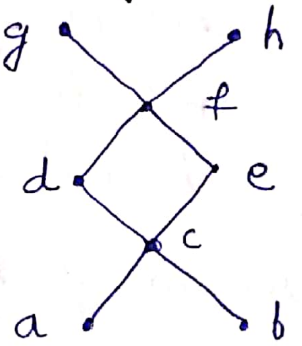
Let $(P, \leq)$ be a poset and let $A \subseteq P$. An element $x \in P$ is said to be a least upper bound or supremum of A if x is a upper bound of A and $x \leq y$ for all upper bound y of A

Greatest lower bound:

Let $(P, \leq)$ be a poset and $A \subseteq P$. An element $x \in P$ is said to be a greatest lower bound or infi (A)

x is a lower bound and $y \leq x$ for all lower bounds y of A

Ex. Let $A = \{a, b, c, d, e, f, g, h\}$. Let $\leq$ on A represented by hasse diagram.



Let $B = \{c, d, e\}$ then B is subset of A. $(B \subseteq A)$

upper bound of $B = \{f, g, h\}$
lower bound of $B = \{a, b, c\}$

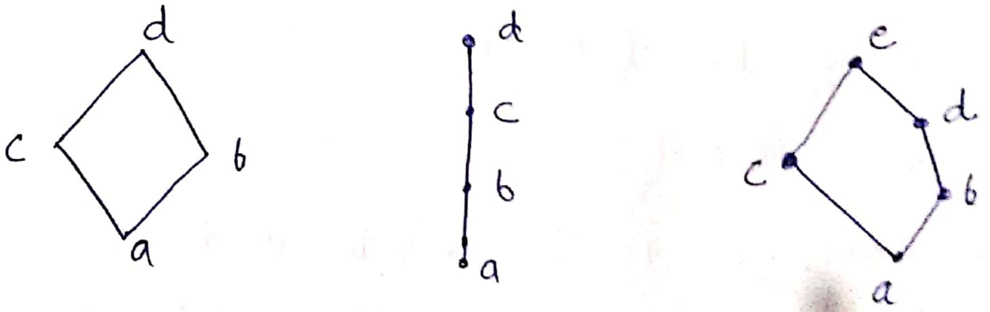
Supremum $= f$ $\quad$ $\sup(B) = f$
Infimum $= c$ $\quad$ $\inf(B) = c$

Lattices :- A lattice is a poset $(L, \leq)$ in which every subset $\{a, b\}$ of two elements of L has a greatest lower bound and a least upper bound.

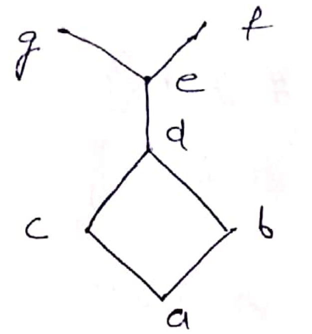
In other words, poset $(L, \leq)$ is a lattice if for every $a, b \in L$ $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in L

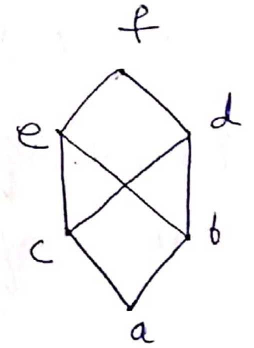
we denote $\sup\{a, b\}$ by $a \vee b$ and call it the join of a and b and $\inf\{a, b\}$ by $a \wedge b$ and call it the meet of a and b

ex.



Which is not lattice





sup $\{f, g\}$ does not exist.

Th:- 1 If $(L, \leq)$ is a lattice with binary operations $\vee$ and $\wedge$, then for element $a, b, c \in L$

(i) $a \leq b \Leftrightarrow a \wedge b = a$

(ii) $a \leq b \Leftrightarrow a \vee b = b$

(iii) $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$ (absorption)

(iv) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ Associative Law

(i) $a \leq b \Leftrightarrow a \wedge b = a$

Let $(L, \leq)$ be lattice

Let $a, b \in L$

we assume that $a \leq b$ and then we shall show $a \wedge b = a$

Since $\leq$ is reflexive

we have $\forall a \in L \Rightarrow a \leq a$ — (1)

also we have $\forall a \in L \Rightarrow a \leq b$ (By assumption)

$a \leq a \wedge b$ —— 2

Thus two result implies that a is lower bound of $\{a, b\}$

But we know that

$a \wedge b = \inf \{a, b\}$

$\therefore a \wedge b \leq a$ —— 3

from eq 2 $a \leq a \wedge b$

eq 2 and eq 3

Hence

$$\boxed{a \wedge b = a}$$

$\Leftarrow$ Let $(L, \leq)$ be a Lattice

Let $a, b \in L$

We assume that $a \wedge b = a$ then

we shall show that $a \leq b$

Since $a \wedge b = a$

$\Rightarrow \inf \{a \wedge b\} = a$

$\Rightarrow a \wedge b \leq b$

$\Rightarrow a \leq b$

Hence prove

(ii) $a \leq b \Leftrightarrow a \vee b = b$

($\Rightarrow$) we assume that $a \vee b = b$
then we shall prove that $a \leq b$

Since $a \vee b = \sup\{a, b\}$

$\therefore a \leq a \vee b$

and $a \vee b = b$

$\therefore a \leq b$

($\Leftarrow$) we assume that $a \leq b$ and then we shall prove that $a \vee b = b$

Since $\leq$ is reflexive

$\therefore \forall b \in L \Rightarrow b \leq b$

$\therefore \sup\{a, b\} \leq b$ (By Assuption)

$a \vee b \leq b$ — ① $\{$ By def^n of join $a \vee b = \sup\{a, b\}$

also we have

$b \leq b$

$b \leq a \vee b$ — ②

from eq ① and ②

$\boxed{a \vee b = b}$

Hence proved.