Th: 2 If $(L, \leq)$ is a lattice with binary operation $\vee$ and $\wedge$, then for arbitary element $a, b, c, d \in L$

(i) $a \leq b$ and $c \leq d \Rightarrow a \wedge c \leq b \wedge d$

(ii) $a \leq b$ and $c \leq d \Rightarrow a \vee c \leq b \vee d$

Proof:- Let $(L, \leq)$ is a lattice and $a, b, c, d \in L$

suppose that $a \leq b$ and $c \leq d$

By defn of meet $(\wedge)$

we have

$$a \wedge c = \inf\{a, c\}$$

$$a \wedge c \leq a \quad \text{and} \quad a \wedge c \leq c$$

Now $a \wedge c \leq a$ and $a \leq b$

$(\because \leq \text{ is transitive})$

$$a \wedge c \leq b$$

and also

$$a \wedge c \leq c \quad \text{and} \quad c \leq d$$

$$a \wedge c \leq d \quad (\because \leq \text{ is transitive})$$

This $a \wedge c$ is lower bound of b and d

$a \wedge c \leq d$

$a \wedge c \leq d$

$\Rightarrow \quad a \wedge c \leq b \wedge d$

(ii) Let $(L, \leq)$ be a lattice

and $a, b, c, d \in L$

Suppose that $a \leq b$ and $c \leq d$

$\Rightarrow a \wedge c \leq b \wedge d$

By defn of join operator $\vee$

we have

$b \vee d = \sup \{b, d\}$

$b \leq b \vee d$ and $d \leq b \vee d$

Now $a \leq b$ and $b \leq b \vee d$

$\because$ ($\leq$ is transitive)

$a \leq b \vee d$

and also

$c \leq d$ and $d \leq b \vee d$

$c \leq b \vee d$ $\because$ ($\leq$ is transitive)

This $b \vee d$ is upper bound of a and c

$\Rightarrow \quad a \vee c \leq b \vee d$

Th:3 for any a and b in a lattice $(L, \leq)$

$a \wedge b \leq a \leq a \vee b$

Proof: By the defn of meet $(\wedge)$ operation we have

$a \wedge b = \inf(a, b)$

$(a \wedge b)$ is a lower bound of $[a, b]$

$(a \wedge b)$ is a lower bound of a

$a \wedge b \leq a$

Also by defn of join $(\vee)$ operation

$a \vee b = \sup \{a, b\}$

$a \vee b$ is a upper bound of a

$a \leq a \vee b$

Note: Lattice is also satisfies the following condition (Law)

(i) Commutative Law

$a \wedge b = b \wedge a$

$a \vee b = b \vee a$

(ii) Associative law

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

$(a \vee b) \vee c = a \vee (b \vee c)$ $\forall a, b, c \in L$

(iii) Abscorpition Law:

$a \wedge (a \vee b) = a$

$a \vee (a \wedge b) = a$ $\forall a, b \in L$

4. $a \vee a = a$ $\quad \forall a \in L$ (Idempotent Law)
$a \wedge a = a$

Th.: Absorption Law

(i) $a \vee (a \wedge b) = a$ $\quad \forall a, b \in L$

(ii) $a \wedge (a \vee b) = a$

Proof: Since $a \vee (a \wedge b)$ is join of a and $a \wedge b$

therefore $a \leq a \vee (a \wedge b)$ — (1)

since $a \leq a$, $a \wedge b \leq a$

By theorem (2)

$\left\{ \begin{array}{l} a \leq b \text{ and } c \leq d \\ \Rightarrow a \wedge c \leq b \wedge d \\ \Rightarrow a \vee c \leq b \vee d \end{array} \right\}$

Since $a \leq a$, $a \wedge b \leq a$

$a \vee (a \wedge b) \leq a \vee a$ — (2)

or

$a \vee (a \wedge b) \leq a$ — (3)

$a \vee a = a$ (Idempo.)

eq (1) and (3)

$\boxed{a \vee (a \wedge b) = a}$